

HYPERBOLIC METRICS, HOMOGENEOUS HOLOMORPHIC FUNCTIONALS AND ZALCMAN'S CONJECTURE

SAMUEL L. KRUSHKAL

ABSTRACT. We show, using the Kobayashi and Carathéodory metrics on special holomorphic disks in the universal Teichmüller space, that a wide class of holomorphic functionals on the space of univalent functions in the disk is maximized by the Koebe function or by its root transforms; their extremality is forced by hyperbolic features. As consequences, this implies the proofs of the famous Zalcman and Bieberbach conjectures.

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1. INTRODUCTION. MAIN THEOREMS

In this paper, we build a bridge linking the hyperbolic Kobayashi and Carathéodory metrics of the universal Teichmüller space and Grunsky inequalities with two famous conjectures in classical geometric complex analysis and prove these conjectures.

1.1. General homogeneous holomorphic functionals. The holomorphic functionals on the classes of univalent functions depending on the Taylor coefficients of these functions play an important role in various geometric and physical applications of complex analysis, for example, in view of their connection with string theory and with a holomorphic extension of the Virasoro algebra. These coefficients reflect the fundamental intrinsic features of conformal maps. Thus estimating them still remains an important problem in geometric function theory.

We consider the univalent functions on the unit disk $\Delta = \{|z| < 1\}$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

These functions form the well-known class S . Their inversions $F_f(z) = 1/f(1/z)$ belong to the class Σ of univalent nonvanishing functions

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots \quad (F(z) \neq 0) \quad (1.1)$$

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on the complementary disk $\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$. Easy computations yield that the coefficients a_n and b_j are related by

$$b_0 + a_2 = 0, \quad b_n + \sum_{j=1}^n b_{n-j} a_{j+1} + a_{n+2} = 0, \quad n = 1, 2, \dots, \quad (1.2)$$

which implies successively the representations of a_n by b_j . One gets

$$a_n = (-1)^{n-1} b_0^{n-1} - (-1)^{n-1} (n-2) b_1 b_0^{n-3} + \text{lower terms with respect to } b_0; \quad (1.3)$$

in particular,

$$\begin{aligned} a_2 &= -b_0, \quad a_3 = -b_1 + b_0^2, \quad a_4 = -b_2 + 2b_1 b_0 - b_0^3, \\ a_5 &= -b_3 + 2b_2 b_0 + b_1^2 - 3b_1 b_0^2 + b_0^4, \\ a_6 &= -b_4 + 2b_3 b_0 + 2b_2 b_1 - 3b_2 b_0^2 - 3b_1^2 b_0 + 4b_1 b_0^3 - b_0^5, \\ a_7 &= b_0^6 - 5b_1 b_0^4 - b_1^3 + 4b_2 b_0^3 + b_2^2 + (6b_1^2 - 3b_3) b_0^2 \\ &\quad + 2b_1 b_3 + (-6b_1 b_2 + 2b_4) b_0 - b_5, \dots \end{aligned}$$

We shall essentially use this connection.

Consider a general holomorphic distortion functional on S of the form

$$J(f) = J(a_2, \dots, a_n; (f^{(\alpha_1)}(z_1)); \dots; (f^{(\alpha_p)}(z_p))), \quad (1.4)$$

where z_1, \dots, z_p are the distinct fixed points in $\Delta \setminus \{0\}$ with assigned orders m_1, \dots, m_p , respectively, $(f^{(\alpha_1)}(z_1)) = f''(z_1), \dots, f^{(m_1)}(z_1)$; $(f^{(\alpha_p)}(z_p)) = f''(z_p), \dots, f^{(m_p)}(z_p)$. Assume that J is a polynomial in all of its variables.

Substituting the expressions of a_j by b_m from (1.2) and calculating $f^{(q)}(z_j)$ in terms of F_f , one obtains a polynomial $\tilde{J}(F)$ of the Taylor coefficients b_0, b_1, \dots, b_{n-2} and of the corresponding derivatives $F_f^{(q)}(\zeta_j)$ at the points $\zeta_j = 1/z_j \in \Delta^* \setminus \{\infty\}$, regarded as a representation of $J(f)$ on the class Σ . Here $q = 2, \dots, m_j$, $j = 1, \dots, p$.

Assume that the functional (1.4) is homogeneous with a degree $d = d(J)$ (depending on n and m_1, \dots, m_p) with respect to homotopy

$$f(z, t) = t^{-1} f(tz) = z + a_2 t + a_3 t^2 + \dots : \Delta \times \overline{\Delta} \rightarrow \mathbb{C}$$

such that $f(z, 0) \equiv z$, $f(z, 1) = f(z)$ so that

$$J(f_t) = t^d J(f).$$

This homotopy is a special case of holomorphic motions with complex parameter t running over the disk Δ . The functional $\tilde{J}(F)$ on Σ admits a similar homogeneity.

The existence of extremal functions of $J(f)$ and $\tilde{J}(F)$ follows from compactness of both classes S and Σ in the topology of locally uniform convergence on Δ and Δ^* , respectively.

1.2. The Zalcman conjecture. There were several classical conjectures about these coefficients. They include the Bieberbach conjecture that in the class S the coefficients are estimated by $|a_n| \leq n$, as well as several other well-known conjectures that imply the Bieberbach conjecture. Most of them have been proved by the de Branges theorem [DB].

In the 1960s, Lawrence Zalcman posed the conjecture that for any $f \in S$ and all $n \geq 3$,

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad (1.5)$$

with equality only for the **Koebe function**

$$\kappa_\theta(z) = \frac{z}{(1 - e^{i\theta}z)^2} = z + \sum_2^\infty n e^{-i(n-1)\theta} z^n, \quad 0 \leq \theta \leq 2\pi, \quad (1.6)$$

which maps the unit disk onto the complement of the ray

$$w = -te^{-i\theta}, \quad \frac{1}{4} \leq t \leq \infty.$$

This remarkable conjecture also implies the Bieberbach conjecture. This still is an intriguing very difficult open problem for all $n > 6$.

The original aim of Zalcman's conjecture was to prove the Bieberbach conjecture using the famous Hayman theorem on the asymptotic growth of coefficients of individual functions, which states that *for each $f \in S$, we have the inequality*

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{n} = \alpha \leq 1,$$

with equality only when $f = \kappa_\theta$; here $\alpha = \lim_{r \rightarrow 1} (1 - r)^2 \max_{|z|=r} |f(z)|$ (see [Ha]).

Indeed, assuming that n is sufficiently large and estimating a_{2n-1} in (1.5) by $|a_{2n-1}| \leq 2n - 1$, one obtains

$$|a_n|^2 \leq (n - 1)^2 + |a_{2n-1}| \leq (n - 1)^2 + 2n - 1 = n^2,$$

which proves the Bieberbach conjecture for this n , and successively for all preceding coefficients.

It was realized almost immediately that the Zalcman conjecture implies the Bieberbach conjecture, and in a very simple fashion, without Hayman's result and without other prior results from the theory of univalent functions.

Note that the case $n = 2$ is rather simple and somewhat exceptional. The inequality $|a_2^2 - a_3| \leq 1$ is well known, but in this case there are two extremal functions of different kinds: the Koebe function $\kappa_\theta(z)$ and the odd function

$$\kappa_{2,\theta}(z) := \sqrt{\kappa_\theta(z^2)} = \sum_{n=0}^\infty e^{in\theta} z^{2n+1}. \quad (1.7)$$

The estimate (1.5) was established for $n = 3$ in [Kr4] and recently for $n = 4, 5, 6$ in [Kr8] (without uniqueness of the extremal function). In [BT], [Ma], this conjecture was proved for certain special subclasses of S .

1.3. Main theorems. It is well known that the Koebe function κ_θ is extremal for many variational problems in the theory of conformal maps (accordingly, its root transforms

$$\kappa_{m,\theta}(z) = \kappa_\theta(z^m)^{1/m} = \frac{z}{(1 - e^{i\theta}z^m)^{2/m}} = z + \frac{2e^{i\theta}}{m} z^{m+1} + \frac{m-2}{m^2} z^{2m+1} + \dots, \quad m = 2, 3, \dots, \quad (1.8)$$

are extremal among the maps with symmetries).

Our first main theorem sheds new light on this phenomenon and provides a large class of functionals maximized by these functions.

Theorem 1.1. *Let $J(f)$ be a homogeneous polynomial functional on S of the form (1.4) whose representation $\tilde{J}(F_f)$ in the class Σ does not contain free terms $c_d b_0^d$ but contains*

nonzero terms with the coefficient b_1 of inversions F_f . Then for all $f \in S$, we have the sharp bound

$$|J(f)| \leq \max_m |J(\kappa_{m,\theta})|, \quad (1.9)$$

and this maximum is attained on some $\kappa_{m_0,\theta}$ ($m_0 \geq 1$). If J has an extremal with

$$b_1 = a_2^2 - a_3 \neq 0, \quad (1.10)$$

then $|b_1| = 1$ and

$$|J(f)| \leq \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\}. \quad (1.11)$$

The assumption (1.10) is equivalent to

$$S_f(0) = - \lim_{z \rightarrow \infty} z^4 S_{F_f}(z) \neq 0.$$

The examples of some well-known functionals, for example, $J(f) = a_2^2 - \alpha a_3$ with $0 < \alpha < 1$ and $J(F_f) = b_m$ ($m > 1$), show that the assumptions on the initial coefficients b_0 and b_1 cannot be omitted.

The Zalcman functional

$$J_n(f) = a_n^2 - a_{2n-1}$$

is a special case of (1.4) with homogeneity degree $2n - 2$. For this functional, we obtain from Theorem 1.1 a complete result proving the Zalcman conjecture.

Theorem 1.2. *For all $f \in S$ and any $n \geq 3$, we have the sharp estimate (1.5), with equality only for $f = \kappa_\theta$.*

As a consequence, one obtains also a new proof of the Bieberbach conjecture.

Theorem 1.1 also provides other new distortion theorems concerning the higher coefficients. These results are presented in the last section.

1.4. It suffices to find the bound of J on functions admitting quasiconformal extensions across the unit circle and make the closure of this set in weak topology determined by locally uniform convergence on Δ . Such functions are naturally connected with the universal Teichmüller space \mathbf{T} . The original functional $J(f)$ is lifted to a holomorphic functional on a fiber space over \mathbf{T} , but its growth is controlled by hyperbolic metrics on the base space \mathbf{T} . Extremity of the Koebe function or of its root transforms is intrinsically connected with the features of these metrics on appropriate geodesic disks.

2. BACKGROUND

We briefly present here certain results underlying the proof of Theorem 1.1. This exposition is adapted to our special cases.

2.1. Universal Teichmüller space. We shall use some deep geometric and potential results concerning the Teichmüller space of the disk, called also the universal Teichmüller space.

(a) The universal Teichmüller space \mathbf{T} is the space of quasiconformal homeomorphisms of the unit circle $S^1 = \partial\Delta$ factorized by Möbius maps. The canonical complex Banach structure on \mathbf{T} is defined by factorization of the ball of the **Beltrami coefficients** (or complex dilatations)

$$\mathbf{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu|_{\Delta^*} = 0, \|\mu\| < 1\}, \quad (2.1)$$

letting $\mu_1, \mu_2 \in \mathbf{Belt}(\Delta)_1$ be equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions to the Beltrami equation $\partial_{\bar{z}}w = \mu\partial_zw$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $S^1 = \partial\Delta^*$ (hence, on $\overline{\Delta^*}$). The equivalence classes $[w^\mu]$ are in one-to-one correspondence with the Schwarzian derivatives

$$S_w(z) := \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2, \quad w = w^\mu|_{\Delta^*}.$$

Note that for each locally univalent function $w(z)$ on a simply connected hyperbolic domain $D \subset \widehat{\mathbb{C}}$ its Schwarzian derivative S_w belongs to the complex Banach space $\mathbf{B}(D)$ of hyperbolically bounded holomorphic functions on D with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_D \lambda_D^{-2}(z)|\varphi(z)|,$$

where $\lambda_D(z)|dz|$ is the hyperbolic metric on D of Gaussian curvature -4 ; hence $\varphi(z) = O(z^{-4})$ as $z \rightarrow \infty$ if $\infty \in D$. In particular, for $D = \Delta$,

$$\lambda_\Delta(z) = 1/(1 - |z|^2). \quad (2.2)$$

The space $\mathbf{B}(D)$ is dual to the Bergman space $A_1(D)$, a subspace of $L_1(D)$ formed by integrable holomorphic functions on D .

The derivatives $S_{w^\mu}(z)$ with $\mu \in \mathbf{Belt}(\Delta)_1$ range over a bounded domain in the space $\mathbf{B} = \mathbf{B}(\Delta^*)$. This domain models the universal Teichmüller space \mathbf{T} , and the defining projection

$$\phi_{\mathbf{T}}(\mu) = S_{w^\mu} : \mathbf{Belt}(\Delta)_1 \rightarrow \mathbf{T}$$

is a holomorphic map from $L_\infty(\Delta)$ to \mathbf{B} . This map is a split submersion, which means that $\phi_{\mathbf{T}}$ has local holomorphic sections (see, e.g., [GL]).

The above definition of \mathbf{T} requires a complete normalization of maps w^μ , which uniquely define the values of w^μ on Δ^* by their Schwarzians. We shall use the condition $w^\mu(0) = 0$ going over from w^μ to the maps

$$w_1^\mu(z) = w^\mu(z) - w^\mu(0) = z - \frac{1}{\pi} \iint_{\Delta} \frac{\partial w^\mu}{\partial \bar{\zeta}} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\xi d\eta \quad (\zeta = \xi + i\eta),$$

which does not reflect on the Schwarzians. We identify the bounded domain in \mathbf{B} filled by $S_{w_1^\mu}$ with the space \mathbf{T} .

The intrinsic **Teichmüller metric** of the space \mathbf{T} is defined by

$$\tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf \{ \log K(w^{\mu_*} \circ (w^{\nu_*})^{-1}) : \mu_* \in \phi_{\mathbf{T}}(\mu), \nu_* \in \phi_{\mathbf{T}}(\nu) \}; \quad (2.3)$$

it is generated by the Finsler structure

$$F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf \{ \|\nu_*/(1 - |\mu|^2)^{-1}\|_\infty : \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu \} \quad (2.4)$$

on the tangent bundle $\mathcal{T}(\mathbf{T}) = \mathbf{T} \times \mathbf{B}$ of \mathbf{T} (here $\mu \in \mathbf{Belt}(\Delta)_1$ and $\nu, \nu_* \in L_\infty(\mathbb{C})$). This structure is locally Lipschitz (cf. [EE]).

The smallest dilatation $k(F) = \inf \|\mu_{\widehat{F}}\|_\infty$ among quasiconformal extensions of $F|_{\Delta^*}$ onto $\widehat{\mathbb{C}}$ is called the **Teichmüller norm** (or dilatation) of F .

The space \mathbf{T} as a complex Banach manifold also has invariant metrics (with respect to its biholomorphic automorphisms); the largest and the smallest invariant metrics are the Kobayashi and the Carathéodory metrics, respectively. Namely, the **Kobayashi metric** $d_{\mathbf{T}}$ on \mathbf{T} is the largest pseudometric d on \mathbf{T} which does not get increased by holomorphic maps $h : \Delta \rightarrow \mathbf{T}$ so that for any two points $\psi_1, \psi_2 \in \mathbf{T}$,

$$d_{\mathbf{T}}(\psi_1, \psi_2) \leq \inf \{d_{\Delta}(0, t) : h(0) = \psi_1, h(t) = \psi_2\},$$

where d_{Δ} is the hyperbolic metric on Δ with the differential form (2.2). This distance is connected with the Teichmüller norm of f by $k(f) = \tanh d_{\mathbf{T}}(\mathbf{0}, S_f)$.

The **Carathéodory distance** between ψ_1 and ψ_2 in \mathbf{T} is

$$c_{\mathbf{T}}(\psi_1, \psi_2) = \sup d_{\Delta}(h(\psi_1), h(\psi_2)),$$

where the supremum is taken over all holomorphic maps $h : \mathbf{T} \rightarrow \Delta$.

The Kobayashi metric is the integrated form of its infinitesimal Finsler metric defined for the points $(\psi, v) \in \mathcal{T}(\mathbf{T})$ by

$$\mathcal{K}_{\mathbf{T}}(\psi, v) = \inf \{1/r : r > 0, h \in \text{Hol}(\Delta_r, \mathbf{T}), h(0) = \psi, h'(0) = v\},$$

where $\text{Hol}(\Delta_r, \mathbf{T})$ denotes the collection of holomorphic maps of the disk $\Delta_r = \{|z| < r\}$ into \mathbf{T} . For the general properties of invariant metrics we refer, for example, to [Di], [Ko].

The Royden-Gardiner theorem states that the Kobayashi and Teichmüller metrics are equal on all Teichmüller spaces (cf. [EKK], [EM], [GL], [Ro]). This fundamental fact underlies many applications of the Teichmüller space theory.

(2) A strengthened version of the Royden-Gardiner theorem for the universal Teichmüller space is given by

Proposition 2.1. [Kr4]. *The infinitesimal Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\psi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space \mathbf{T} is logarithmically plurisubharmonic in $\psi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\psi, v)$ on $\mathcal{T}(\mathbf{T})$ generating the Teichmüller metric of \mathbf{T} and has constant holomorphic sectional curvature -4 .*

It implies that the Teichmüller distance $\tau_{\mathbf{T}}(\varphi, \psi)$ is logarithmically plurisubharmonic in each of its variables and hence the pluricomplex Green function of the space \mathbf{T} equals

$$g_{\mathbf{T}}(\varphi, \psi) = \log \tanh \tau_{\mathbf{T}}(\varphi, \psi) = \log k(\varphi, \psi), \quad (2.5)$$

where $k(\varphi, \psi)$ denotes the extremal dilatation of quasiconformal maps determining the Teichmüller distance between the points φ and ψ in \mathbf{T} .

Recall that the **pluricomplex Green function** $g_D(x, y)$ of a domain D in a complex Banach manifold X with pole y is defined by $g_D(x, y) = \sup u_y(x)$ ($x, y \in D$) and followed by upper regularization $v^*(x) = \limsup_{x' \rightarrow x} v(x')$, taking the supremum over all plurisubharmonic functions $u_y(x) : D \rightarrow [-\infty, 0)$ such that

$$u_y(x) = \log \|x - y\|_X + O(1)$$

in a neighborhood of the pole y . Here $\|\cdot\|_X$ denotes the norm on the space modeling X , and the remainder term $O(1)$ is bounded from above. The Green function $g_D(x, y)$ is a maximal plurisubharmonic function on $D \setminus \{y\}$ (unless it is identically $-\infty$).

(c) The assumption $F^\mu(0) = 0$ for quasiconformal extensions of a function $F \in \Sigma^0$ made above ensures nonvanishing F in Δ^* (and $f(z) = 1/F(1/z) \in S$).

In addition, completely normalized maps $F^\mu(z)$ are holomorphic functions of their Beltrami coefficients $\mu \in \mathbf{Belt}(\Delta)_1$ as well as of their Schwarzians. The same holds for the coefficients b_m of F^μ .

The Schwarzian equations $S_w = \varphi$ for $F \in \Sigma$ and for its inversion $f \in S$ and the Beltrami equation for quasiconformal extensions of these functions determine their solutions up to linear transformations, which for F are the translations $w \mapsto w + b_0$ and for f have the form

$$w \mapsto w/(1 - \alpha w) = w + \alpha w^2 + \dots$$

with α determined by the coefficients a_2 . The admissible values of $b_0 = -a_2$, which are the free terms of corresponding $F_1 \in \Sigma$ having the same Schwarzian, are only those which range over the closed domain $\widehat{\mathbb{C}} \setminus F(\Delta^*)$.

Since the original normalization of the functions from Σ and S includes only two conditions, the initial functional J_n must be considered on the **fiber space** $\mathcal{F}(\mathbf{T})$ over \mathbf{T} , which is modeled as a bounded domain in the space $\mathbf{B} \times \Delta(0, 2)$, whose points are the pairs (φ, a) where φ are the Schwarzians of $F \in \Sigma^0$ with $F(0) = 0$ and a are equal to the second coefficients a_2 of their inversions in S . The defining projection of this space

$$\pi_{\mathcal{F}} : (S_{F^\mu}, a_2^\mu) \rightarrow S_{F^\mu}$$

is a holomorphic split submersion. The fibers $\pi_{\mathcal{F}}^{-1}(\varphi)$ over the base points $\varphi = S_F$ coincide with the complementary domains $\widehat{\mathbb{C}} \setminus \overline{F(\Delta^*)}$, giving the admissible values of a_2 .

Note also that the space $\mathcal{F}(\mathbf{T})$ is biholomorphically isomorphic to the Bers fiber space over \mathbf{T} (both fiber spaces have the same base and differ only by an additional normalization of maps $F \in \Sigma$ with a given Schwarzian $\varphi = S_F$) and thus is isomorphic to the Teichmüller space of the punctured disk $\Delta \setminus \{0\}$ (cf. [Be2]).

2.2. The Grunsky operator. The complex geometry of the universal Teichmüller space is closely connected with the Grunsky inequalities technique which arose from investigating the univalence problem in [Gr].

Any function $F \in \Sigma$ (and similarly any $f \in S$) determines its Grunsky operator (matrix) $\mathcal{G}_F = (\alpha_{mn}(F))$, where the Grunsky coefficients α_{mn} are determined by the expansion

$$\log \frac{F(z) - F(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

with the principal branch of logarithmic function, satisfy the inequalities

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq k. \quad (2.6)$$

Here $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \left(\sum_1^{\infty} |x_n|^2 \right)^{1/2}$, and $k = k(F) \leq 1$ is the Teichmüller norm of F (cf. [Gr], [Ku1]). The

quantity

$$\varkappa(F) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : \mathbf{x} = (x_n) \in S(l^2) \right\} \leq 1$$

is called the **Grunsky norm** of F . It equals the norm of \mathcal{G}_F regarded as a linear operator $l^2 \rightarrow l^2$.

The functions with $\varkappa(F) = k(F)$ play a crucial role in applications of Grunsky inequalities; however, the set of S_F , on which $\varkappa(F) < k(F)$, is open and dense in \mathbf{T} . One of the underlying facts in applications is the following result.

Proposition 2.2. *The equality $\varkappa(F) = k(F)$ for $f \in \Sigma^0$ holds if and only if the function F is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_0 satisfying the condition $\sup |\langle \mu_0, \psi \rangle_{\Delta}| = \|\mu_0\|_{\infty}$, where the supremum is taken over holomorphic functions $\psi \in A_1^2(\Delta)$ with $\|\varphi\|_{A_1(\Delta)} = 1$, where*

$$A_1^2 = \{\psi \in A_1(\Delta) : \psi = \omega^2 \text{ with } \omega \text{ holomorphic on } \Delta\}.$$

In addition, if the equivalence class $[F]$ contains a frame map, i.e., is a Strebel point (see Section 2.4), then the restriction of μ_0 onto the disk Δ must be of the form

$$\mu_0(z) = k|\psi_0(z)|/\psi_0(z) \quad \text{with } \psi_0 \in A_1^2. \quad (2.7)$$

The proof of this proposition is given in [Kr2], [Kr7]. It relies on the fact that the Grunsky coefficients $\alpha_{mn}(S_F)$ generate the holomorphic functions

$$h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(\varphi) x_m x_n, \quad (2.8)$$

where $\varphi = S_f$ and $\mathbf{x} = (x_n)$ are the points of the sphere $S(l^2)$, mapping the universal Teichmüller space \mathbf{T} into the unit disk Δ . The restrictions of these functions to the disk $\{\phi_{\mathbf{T}}(s\mu_0)\}$ determine the Carathéodory distance between the points $S_{fs\mu_0}$ and the origin, which by (2.6) equals the Teichmüller distance.

In a special case, when the curve $F(S^1)$ is analytic, the equality (2.7) was obtained by a different method in [Ku2].

In particular, the maps

$$F_{m,t}(z) = \frac{1}{\kappa_{m,t}(1/z)} = z \left(1 - \frac{t}{z^{m+1}}\right)^{2/(m+1)} = z - \frac{2t}{m+1} \frac{1}{z^m} + \dots, \quad |t| \leq 1, \quad (2.9)$$

whose extremal extensions to \mathbb{C} have Beltrami coefficients

$$\mu_{F_{m,t}}(z) = t|z|^{m-1}/z^{m-1} \quad \text{for } |z| < 1,$$

satisfy $\varkappa(F_{m,t}) = k(F_{m,t})$ for odd $m \geq 1$ and $\varkappa(F_{m,t}) < k(F_{m,t})$ for even $m \geq 0$.

Note that holomorphy of the functions (2.8) is a consequence of the fact that the Grunsky coefficients α_{mn} are polynomials of the initial coefficients b_1, \dots, b_{m+n-1} of F combined with the well-known inequality (cf. [Po, p. 61]) : for any $1 \leq p \leq M$, $1 \leq q \leq N$,

$$\left| \sum_{m=p}^M \sum_{n=q}^N \sqrt{mn} \alpha_{mn} x_m x_n \right|^2 \leq \sum_{m=p}^M |x_m|^2 \sum_{n=q}^N |x_n|^2.$$

We mention also that both Teichmüller and Grunsky norms are continuous logarithmically plurisubharmonic functions on \mathbf{T} (see [Kr6], [Sh]) and that, by a theorem of Pommerenke and Zhuravlev, any $F \in \Sigma$ with $\kappa(F) \leq k < 1$ has k_1 -quasiconformal extensions to $\widehat{\mathbb{C}}$ with $k_1 = k_1(k) \geq k$ (see [Po]; [KK1, pp. 82-84], [Zh]).

2.3. A holomorphic homotopy of univalent function. Similarly to the functions in S , one can define for each $F \in \Sigma$ with expansion (1.1) the complex homotopy

$$F_t(z) = tF\left(\frac{z}{t}\right) = z + b_0t + b_1t^2z^{-1} + b_2t^3z^{-2} + \dots : \Delta^* \times \Delta \rightarrow \widehat{\mathbb{C}} \quad (2.10)$$

so that $F_0(z) \equiv z$. This implies

$$S_{F_t}(z) = t^{-2}S_F(t^{-1}z),$$

and moreover, this point-wise map determines a holomorphic map

$$h_F(t) = S_{F_t}(\cdot) : \Delta \rightarrow \mathbf{B} \quad (2.11)$$

(see, e.g. [Kr3]). The corresponding **homotopy disk**

$$\Delta(S_F) = h_F(\Delta) \subset \mathbf{T} \quad (2.12)$$

is holomorphic at noncritical points of maps (2.11). These disks foliate the space \mathbf{T} (and the set Σ).

The dilatations of the homotopy maps are estimated by

Proposition 2.3. [Kr3] (a) *Each homotopy map F_t of $F \in \Sigma$ admits k -quasiconformal extension to the complex sphere $\widehat{\mathbb{C}}$ with $k \leq |t|^2$. The bound $k(F_t) \leq |t|^2$ is sharp and occurs only for the maps*

$$F_{b_0, b_1}(z) = z + b_0 + b_1z^{-1}, \quad |b_1| = 1,$$

whose homotopy maps

$$F_{b_0, b_1 t^2}(z) = z + b_0t + b_1t^2z^{-1} \quad (2.13)$$

have the affine extensions $\widehat{F}_{b_0, b_1 t^2}(z) = z + b_0t + b_1t^2\bar{z}$ onto Δ .

(b) *If $F(z) = z + b_0 + b_m z^{-m} + b_{m+1} z^{-(m+1)} + \dots$ ($b_m \neq 0$) for some integer $m > 1$, then the minimal dilatation of extensions of F_t is estimated by $k(F_t) \leq |t|^{m+1}$; this bound also is sharp.*

In the second case,

$$h_F(0) = h'_F(0) = \dots = h_F^{(m)}(0) = \mathbf{0}, \quad h_F^{(m+1)}(0) \neq \mathbf{0},$$

and due to [KK2],

$$k(F_t) = \frac{m+1}{2} |b_m| |t|^{m+1} + O(t^{m+2}), \quad t \rightarrow 0. \quad (2.14)$$

This bound is sharp.

The simplest holomorphic disks in Σ^0 are the images of Teichmüller **extremal disks**

$$\Delta(\psi) = \{\phi_{\mathbf{T}}(t\mu_0) : t \in \Delta\} \subset \mathbf{T}$$

formed by functions F whose extremal extensions onto Δ (with minimal dilatation) have Beltrami coefficients $t\mu_0$ with

$$\mu_0(z) = |\psi(z)|/\psi(z), \quad (2.15)$$

where ψ is a holomorphic function from $L_1(\Delta)$. Such an extension is unique (up to a constant factor of ψ and normalization of F). The Teichmüller disks foliate dense subsets in \mathbf{T} and

Σ^0 . Note also that the homotopy function F_t has such extremal extension for each $t \in \Delta$ (cf. [GL], [St]) and that for any function (2.9) its homotopy disk $\Delta(S_{F_m})$ is of Teichmüller type.

3. PROOF OF THEOREM 1.1

(a) Assume that there exists an extremal of $J(f)$ satisfying the assumption (1.10), and consider first the functions $f \in S$, for which this inequality is fulfilled. The set of the corresponding Schwarzians S_{F_f} is dense in \mathbf{T} , and their maps (2.12) satisfy

$$h_F(0) = h'_F(0) = \mathbf{0}, \quad h''_F(0) \neq \mathbf{0}.$$

Using the relations (1.2), we represent J as a polynomial functional on Σ , which takes the form

$$J(f) = \tilde{J}(F_f) = \tilde{J}_n(b_0, b_1, \dots, b_{2n-3}; F''_f(\zeta_1), \dots, F^{(m_1)}_f(\zeta_1); \dots, F''_f(\zeta_p), \dots, F^{(m_p)}_f(\zeta_p)) \quad (3.1)$$

(where $b_0 = -a_2$ and $\zeta_j = 1/z_j$) and put

$$\tilde{J}^0(F_f) = \frac{\tilde{J}(F_f)}{M(J)} \quad \text{with} \quad M(J) = \max_S |J(f)|$$

to have a holomorphic map $\mathcal{F}(\mathbf{T}) \rightarrow \Delta$.

As was mentioned above, the admissible values of $-b_0 = a_2$ for $F(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0$ with $F(0) = 0$ range over the closed domain $F(\overline{\Delta}) = \widehat{\mathbb{C}} \setminus F(\Delta^*)$. In view of the maximum principle, it suffices to use only the boundary points of domains $F(\Delta^*)$.

We select on the unit circle S^1 a dense subset

$$e = \{Z_1, Z_2, \dots, Z_m, \dots\}.$$

Then the images $F(Z_1), \dots, F(Z_m), \dots$ for $F \in \Sigma^0$ (with normalization indicated above) generate a sequence of holomorphic maps

$$g_m(\varphi) = \tilde{J}^0(-F(Z_m), b_1(\varphi), \dots, b_{2n-3}(\varphi); \{F^{(m_j)}(\zeta_j(\varphi))\}) : \mathbf{T} \rightarrow \Delta, \quad m = 1, 2, \dots \quad (3.2)$$

where $\{F^{(m_j)}(\zeta_j(\varphi))\}$ denotes the collection

$$F''(\zeta_1), \dots, F^{(m_1)}(\zeta_1); \dots; F''(\zeta_p), \dots, F^{(m_p)}(\zeta_p).$$

The upper envelope

$$\mathcal{J}(S_F) = \sup_m |g_m(S_f)|,$$

followed by its upper semicontinuous regularization $\mathcal{J}(\varphi) = \limsup_{\varphi^* \rightarrow \varphi} \mathcal{J}(\varphi^*)$, yields a logarithmically plurisubharmonic functional $\mathbf{T} \rightarrow [0, 1)$ so that

$$\sup_{\mathbf{T}} \mathcal{J}(S_F) = \sup_{\Sigma^0} |\tilde{J}^0(F)| = \max_S |J(f)|/M(J).$$

One may assume that the degree d of J (hence of \mathcal{J}) is even, replacing, if needed, this functional by its square J^2 .

In accordance with normalization $F_f(0) = 0$ of extended maps, we pick in (2.13)

$$F_{0,b_1}(z) = z + b_1 z^{-1}.$$

The extremal extension of this map onto $\overline{\Delta}$ is $z + b_1 \bar{z}$. Now we split every homotopy function F_t of $F = F_f$ by

$$F_t(z) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \dots = F_{0,b_1 t^2}(z) + h(z, t).$$

For sufficiently small $|t|$, the remainder h is estimated by $h(z, t) = O(t^3)$ uniformly in z for all $|z| \geq 1$. Then, by the well-known properties of Schwarzians, we have

$$S_{F_t}(z) = S_{F_{b_0, b_1 t^2}}(z) + \omega(z, t) = S_{F_{0, b_1 t^2}}(z) + \omega(z, t),$$

where the remainder ω is uniquely determined by the chain rule

$$S_{w_1 \circ w}(z) = (S_{w_1} \circ w)(w')^2(z) + S_w(z),$$

and is estimated in the norm of \mathbf{B} by

$$\|\omega(\cdot, t)\|_{\mathbf{B}} = O(t^3), \quad t \rightarrow 0; \quad (3.3)$$

this estimate is uniform for $|t| < t_0$ (cf., e.g. [Be1], [Kr1]).

Then, in view of holomorphy, every map (3.2) satisfies for small $|t|$,

$$g_m(S_{F_t}) = g_m(S_{F_{0, b_1 t^2}}) + O(t^{d+1}), \quad (3.4)$$

where the term $O(t^{d+1})$ is estimated uniformly for all n . Thus

$$\mathcal{J}(S_{F_t}) = \mathcal{J}(S_{F_{0, b_1 t^2}}) + O(t^{d+1}) \quad t \rightarrow 0.$$

Combining with the equality

$$\mathcal{J}(S_{F_t}) = t^d \mathcal{J}(S_F) + O(t^{d+1}),$$

which follows from d -homogeneity of the functionals $\tilde{\mathcal{J}}$ and \mathcal{J} , one obtains

$$t^d \mathcal{J}(S_F) = \mathcal{J}(S_{F_{0, b_1 t^2}}) + O(t^{d+1}). \quad (3.5)$$

Our goal now is to evaluate the values of $\mathcal{J}(S_{F_{0, b_1 t^2}})$. We first prove the following lemma estimating the distortion on geodesic disks in generic complex Banach manifolds X . Denote the Kobayashi and Carathéodory metrics on X by d_X and c_X and assume that there is a holomorphic map $h : \Delta \rightarrow X$ such that

$$d_{\Delta}(t_1, t_2) = c_X(h(t_1), h(t_2)) = d_X(h(t_1), h(t_2))$$

for any two distinct points $t_1, t_2 \in \Delta$. Then $h(\Delta)$ is a holomorphically embedded disk, geodesic for both metrics c_X and d_X . Such isometries between the hyperbolic metrics on Δ and X are regarded as the *complex geodesics* (cf. [Ve]).

Lemma 3.1. *Let X be a complex Banach manifold with complete Kobayashi and Carathéodory metrics, and let $h : \Delta \rightarrow X$ be a complex geodesic. If the restriction of a holomorphic map $X \rightarrow \Delta$ to the disk $D = h(\Delta)$ has at $x_0 = h(0)$ zero of order $p \geq 1$, then for all $t \in \Delta$,*

$$|j \circ h(t)| \leq \tanh d_X(h(t^p), \mathbf{0}). \quad (3.6)$$

The equality (even for one $t \neq 0$) only occurs for defining functions of the Carathéodory distance between the points x_0 and $x \in D$, then

$$d_{\Delta}(j \circ h(t^p), j \circ h(0)) = c_X(h_0(t^p), h_0(0)) \quad \text{for all } t \in \Delta.$$

Proof. First recall the Schwarz lemma for subharmonic functions (we only need its simplest version).

Lemma 3.2. *Let a function $u(t) : \Delta \rightarrow [0, 1)$ be logarithmically subharmonic in the disk Δ and such that the ratio $u(t)/|t|^m$ is bounded in a neighborhood of the origin for some $m \geq 1$. Then*

$$u(t) \leq |t|^m \quad \text{for all } t \in \Delta \quad (3.7)$$

and

$$\limsup_{|t| \rightarrow 0} \frac{u(t)}{|t|^m} \leq 1. \quad (3.8)$$

Equality in (3.7), even for one $t_0 \neq 0$, or in (3.8), can only hold for the function $u(t) = |t|^m$.

The assumption on metrics d_X and c_X yields that the pluricomplex Green function $g_X(x, y)$ of X with a pole at y cannot be equal identically $-\infty$ and that for any pair of points $x, y \in X$ with equal Kobayashi and Carathéodory distances,

$$g_X(x, y) = \log \tanh d_X(x, y) = \log \tanh c_X(x, y); \quad (3.9)$$

in addition, $\lim g_X(x, y) = 0$ as x tends to infinity (boundary of X), for any fixed $y \in X$. From (3.9) and Lemma 3.2,

$$g_X(h(t^p), \mathbf{0}) = \log |t|^p \quad \text{for all } |t| < 1.$$

The function $|j \circ h(t)|$ is logarithmically subharmonic on Δ and has zero of order p at the origin, hence by the same Schwarz's lemma,

$$\log |j \circ h(t)| \leq \log |t|^p,$$

which implies the estimate (3.6), completing the proof of Lemma 3.1.

We proceed to the proof of Theorem 1.1 and denote by s the canonical complex parameter on the Teichmüller disks in $\mathcal{F}(\mathbf{T})$ generated by admissible (that is, nonvanishing on Δ^*) functions

$$F_{b_0, s}(z) = z + b_0 + sz^{-1},$$

whose extremal extensions onto $\overline{\Delta}$ are the affine maps $z \mapsto z + b_0 + s\bar{z}$. These disks cover $\Delta(S_{F_0, s}) \subset \mathbf{T}$. If such $F_{b_0, s}$ is admissible only for $|s| < s_0 < 1$, one can reparametrize it using the parameter $\sigma = s/s_0$ which runs over the unit disk. For each b_0 ,

$$d_{\mathbf{T}_1}(\mathbf{0}, (S_{F_{b_0, s}}, b_0)) = d_{\mathbf{T}}(\mathbf{0}, S_{F_0, s}); \quad (3.10)$$

in addition, every map

$$\sigma \mapsto (S_{F_{b_0, \sigma}}, b_0 \sigma), \quad \sigma \in \mathbf{T},$$

determines a complex geodesics in the space $\mathcal{F}(\mathbf{T})$.

Since by (2.14) the parameters s and t are related near the origin by

$$s = b_1 t^2 + O(t^3) \quad \text{as } t \rightarrow 0$$

and $b_1 \neq 0$, it follows from (3.5) that the restrictions of the functions (3.4) to any disk $\Delta(S_{F_{b_0, s}})$ have zero of order $d/2$ at the origin. Lemma 3.1 implies the bound

$$|g_m(S_{F_{b_0, s}})| \leq [\tanh d_{\mathbf{T}}(S_{F_{b_0, s}}, \mathbf{0})]^{d/2} = |s|^{d/2},$$

or equivalently,

$$|g_m(S_{F_{0, b_1 t^2}})| \leq |b_1|^{d/2} |t|^d + O(|t|^{d+1}), \quad t \rightarrow 0. \quad (3.11)$$

Note that by (3.3) the remainder in (3.11) does not depend on m . Therefore,

$$\mathcal{J}(S_{F_{0, b_1 t^2}}) = \sup_m |g_m(S_{F_{0, b_1 t^2}})| \leq |b_1|^{d/2} |t|^d + O(|t|^{d+1}). \quad (3.12)$$

The relations (3.5) and (3.12) imply the equality

$$\mathcal{J}(S_F) \leq |b_1|^{d/2} = \left(\frac{|S_f(0)|}{6} \right)^{d/2} \leq 1. \quad (3.13)$$

Now observe that if the original functional $J(f)$ has an extremal $f_0(z) = z + \sum_2^\infty a_n^0 z^n$, for which the assumption (1.10) is fulfilled, then the value of the left-hand side in (3.21) on this function must be equal to 1, because in this case,

$$\frac{|\tilde{J}(S_{F_0}, a_2^0)|}{M(J)} = \mathcal{J}(S_{F_0}) = 1,$$

thus

$$\frac{1}{6}|S_{f_0}(0)| = |(a_2^0)^2 - a_3^0| = 1.$$

As was mentioned, such equalities can only occur when f_0 either is the Koebe function κ_θ or it coincides with the odd function $\kappa_{2,\theta}$ defined by (1.6). In addition, the extremality of f_0 implies

$$|J(f_0)| = M(J) = \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\}.$$

(b) The functions $f \in S$ with $S_f(0) = 0$ omitted above can be approximated (in \mathbf{B} -norm) by f with $S_f(0) \neq 0$ by applying special quasiconformal deformations of the plane given by the following lemma from [Kr1, Ch. 4]. This lemma softens the strongest rigidity of conformal maps.

Lemma 3.3. *In a finitely connected domain $D \subset \widehat{\mathbb{C}}$, let there be selected a set E of positive two-dimensional Lebesgue measure and the distinct finite points z_1, \dots, z_n with assigned non-negative integers $\alpha_1, \dots, \alpha_n$, respectively, so that $\alpha_j = 0$ for $z_j \in E$. Then, for sufficiently small $\varepsilon > 0$ and $\varepsilon \in (0, \varepsilon_0)$, for any given system of numbers $\{w_{sj}\}$, $s = 0, 1, \dots, \alpha_j$, $j = 1, \dots, n$, such that $w_{0j} \in D$,*

$$|w_{0j} - z_j| \leq \varepsilon, \quad |w_{1j} - 1| \leq \varepsilon, \quad |w_{sj}| \leq \varepsilon \quad (s = 2, \dots, \alpha_j, \quad j = 1, \dots, n),$$

there exists a quasiconformal automorphism h_ε of the domain D , which is conformal on the set $D \setminus E$ and satisfies $h_\varepsilon^{(s)}(z_j) = w_{sj}$ for all $s = 0, 1, \dots, \alpha_j$ and $j = 1, \dots, n$, with dilatation $\|\mu_{h_\varepsilon}\|_\infty \leq M\varepsilon$. The constants ε_0 and M depend only on D , E and the vectors (z_1, \dots, z_n) , $(\alpha_1, \dots, \alpha_n)$.

If the boundary Γ of domain D is Jordan or belongs to the class $C^{l,\alpha}$, where $0 < \alpha < 1$ and $l \geq 1$, one can take $z_j \in \Gamma$ with $\alpha_j = 0$ or $\alpha_j \leq l$, respectively.

Now, let $f \in S^0$ have coefficients a_2 and a_3 related by $a_3 = a_2^2$, i.e., $b_1(f) := b_1(F_f) = 0$. Since $f(\Delta^*)$ is a domain, one can take there a set E of positive measure and construct by Lemma 3.5 for a sequence $\varepsilon_n \rightarrow 0$ such variations $h_n = h_{\varepsilon_n}$ of f that for each n ,

$$b_1(h_n \circ f) = b_1(f) + O(\varepsilon_n) \neq 0, \quad |J(h_n \circ f)| = |J(f)| + O(\varepsilon_n) > |J(f)|.$$

Since, by the previous step,

$$|J(h_n \circ f)| \leq \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\},$$

the same estimate will hold also for f .

(c) Finally, consider the case when J has no extremals f_0 satisfying (1.10), and hence any extremal inversion F_{f_0} is of the form

$$F(z) = z + b_0 + b_m z^{-m} + b_{m+1} z^{-(m+1)} + \dots \quad (b_m \neq 0; |z| > 1) \quad (3.14)$$

with $m > 1$. If $m+1$ does not divide $d = d(J)$, we consider the functional J^{m+1} , which is $d(m+1)$ -homogeneous; otherwise one can use J .

Let us show that one can apply to J^{m+1} the above arguments, replacing $F_{0,b_1 t^2}$ by the corresponding function $F_{m,t}$ represented by (2.9). Its Schwarzian relates to S_{F_t} by

$$S_{F_t} = S_{F_{m,t}} + O(t^{m+1}), \quad t \rightarrow 0.$$

If m is odd, one immediately derives from Proposition 2.2 (applied to $\mu_{F_{m,t}}(z) = t|z|^{m-1}/z^{m-1}$) that

$$c_{\mathbf{T}}(\mathbf{0}, S_{F_{m,t}}) = d_{\mathbf{T}}(\mathbf{0}, S_{F_{m,t}}) = \tau_{\mathbf{T}}(\mathbf{0}, S_{F_{m,t}}) = \tanh^{-1} \left(\frac{m+1}{2} |b_m| |t|^{m+1} \right). \quad (3.15)$$

If m is even, we consider the map

$$F_2(z) = F(z^2)^{1/2} = z + \frac{b_0}{2} \frac{1}{z} + \frac{b_m}{2} \frac{1}{z^{2m-1}} + \dots,$$

which is well defined, since $F(0) = 0$, and represents an odd function symmetric with respect to the origin. Denote the Taylor coefficients of F_2 by $b_j^{(2)}$ and let $\alpha_{mn}^{(2)}(F) = \alpha_{mn}(F_2)$. Squaring

$$\mathcal{R}_2 : F(z) \mapsto F(z^2)^{1/2}$$

transforms the quadratic differentials $\psi = \psi(z)dz^2$ in Δ into $\mathcal{R}_2^* \psi = \psi_r(z^2)4z^2 dz^2$, which have zero of even order at the origin. Thus one can apply Proposition 2.2 to F_2 , using instead of (2.8) the functions

$$h_{2,\mathbf{x}}(\varphi) = \sum_{p,q=1}^{\infty} \sqrt{pq} \alpha_{pq}^{(2)}(\varphi) x_p x_q : \mathbf{T} \rightarrow \Delta. \quad (3.16)$$

Indeed, all maps f and F_f are completely normalized and their Beltrami coefficients and Schwarzians are related by

$$\mu_F(z) = \mu_f(1/z)z^2/\bar{z}^2, \quad S_F(z) = -S_f(1/z)z^{-2}.$$

Applying the Cauchy formula for derivatives of holomorphic functions, one derives that the Taylor coefficients a_n , $n \geq 2$, and b_j , $j \geq 0$, depend holomorphically on Beltrami coefficients $\mu_F \in \mathbf{Belt}(\Delta)_1$ and on Schwarzians $S_F \in \mathbf{T}$.

On the other hand, each of the coefficients $b_j^{(2)}$ and $\alpha_{pq}^{(2)}$ of F_2 is represented as a polynomial of a finite number of initial coefficients b_0, b_1, \dots, b_s of the original function F (noting that the free term b_0 is uniquely determined by assumption $F(0) = 0$). Thus the maps (3.16) also depend holomorphically on μ_F and S_F . Finally, the transform \mathcal{R}_2 preserves quasiconformal dilatations.

These properties imply that $S_{F_{m,t}}$ satisfies all equalities in (3.15) and hence ranges in a geodesic disk in \mathbf{T} .

Combining with (2.14), one now obtains instead of (3.13) the bound

$$\mathcal{J}(S_F)^{m+1} \leq \left(\frac{m+1}{2} |b_m| \right)^d \leq 1,$$

or

$$\mathcal{J}(S_F) \leq \left(\frac{m+1}{2} |b_m| \right)^{d/(m+1)} \leq 1. \quad (3.17)$$

To examine the case of equality in (3.17), observe that since $\mathcal{J}(S_{F_{f_0}}) = 1$ for any extremal f_0 , (3.17) yields (denoting the coefficients of F_{f_0} by b_j^0)

$$\frac{m+1}{2} |b_m^0| = 1.$$

The functions (3.13) with $m > 1$ satisfy $b_1 = \dots = b_{m-1} = 0$, which allows us to estimate b_m using the well-known inequalities of Golusin and Jenkins (see [Go, Ch. XI], [Je]). These inequalities hold for a more general univalent $F(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$, $|z| > 1$ (but not for all $F \in \Sigma$), and in our special case imply the bound

$$|b_m| \leq 2/(m+1).$$

Moreover, the equality here only occurs for the function (2.9) with $|t| = 1$ (and its admissible translations $F_{m,t}(z) + c$), which also implies $M(J) = |J(\kappa_{m,\theta})|$.

We have established that any extremal function f_0 maximizing $|J(f)|$ must be of the form (1.8). The theorem is proved.

Remark. Lemma 3.1 is interesting for its own sake and has other applications. Thus we also present a different proof which does not involve Green's function and Lemma 3.2. By assumption,

$$g \circ h(t) = c_p t^p + c_{n+1} t^{p+1} + \dots \quad (c_p \neq 0).$$

Let $\{g_m\}$ be a maximizing sequence for the Carathéodory distance $c_X(x_0, x)$ with $g_m(x_0) = 0$, where $x \in D = h(\Delta)$ is distinct from x_0 , i.e.,

$$c_X(x_0, x) = \lim_{m \rightarrow \infty} d_{\Delta}(0, g_m(x)).$$

The restrictions of g_m to D are convergent locally uniformly on D to a holomorphic function g_0 on this disk with $g_0(x_0) = 0$ so that $d_{\Delta}(0, g_0(x)) = c_X(x_0, x)$, $x \in D$. Since both metrics c_X and d_X are hyperbolic isometries between Δ and D , the map $g_0 \circ h$ is the identity on Δ ; hence, $g_0^p \circ h(t) = t^p$ for all $t \in \Delta$, and

$$|j \circ h(t)| \leq |g_0^p \circ h(t)|. \quad t \in \Delta.$$

Together with the equality $g_0 \circ h(t^p) = t^p$, $t \in \Delta$, this implies (3.6). The case of equality in (3.6) follows from Schwarz's lemma for holomorphic functions with zero of a prescribed order at the origin.

4. PROOF OF THEOREM 1.2

Note that from (1.3),

$$a_n^2 - a_{2n-1} = b_1 b_0^{2n-4} + \text{lower terms with respect to } b_0.$$

We have to show that for all $m > 1$,

$$|J_n(\kappa_{m,\theta})| < J_n(\kappa), \quad (4.1)$$

then Theorem 1.1 implies that only the Koebe function is extremal for Zalcman's functional.

This inequality is trivial for $m = 2$, because the series (1.7) yields

$$|J_n(\kappa_{2,\theta})| \leq 2 < J_n(\kappa).$$

For $m \geq 3$, we apply a result of [Kr5] solving the coefficient problem for univalent functions with quasiconformal extensions with small dilatations. Denote by $S_\infty(k)$ the subclass of S^0 consisting of the functions $f \in S$ having k' -quasiconformal extensions \hat{f} to $\hat{\mathbb{C}}$ ($k' \leq k$), which satisfy $\hat{f}(\infty) = \infty$, and let

$$f_{1,t}(z) = \frac{z}{(1 - ktz)^2}, \quad |z| < 1, \quad |t| = 1.$$

Proposition 4.1. [Kr5] *For all $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\infty(k)$ and all $k \leq 1/(n^2 + 1)$, we have the sharp bound*

$$|a_n| \leq \frac{2k}{n-1}, \quad (4.2)$$

with equality only for the functions

$$f_{n-1,t}(z) = f_{1,t}(z^{n-1})^{1/(n-1)} = z + \frac{2kt}{n-1} z^n + \dots, \quad n = 3, 4, \dots \quad (4.3)$$

Note that every function (4.3) admits a quasiconformal extension onto Δ^* with Beltrami coefficient $\mu_n(z) = t|z|^{n+1}/z^{n+1}$ and $\hat{f}_{n-1,t}(\infty) = \infty$. Accordingly, $F_{n-1,t}(z) = 1/f_{n-1,t}(1/z) \in \Sigma^0$ admits a quasiconformal extension onto the unit disk with $F_{n-1,t}(0) = 0$ and $\mu_{F_{n-1,t}}(z) = t|z|^{n-1}/z^{n-1}$ for $|z| < 1$. Another essential point is that for any function

$$F_{n-1}(z) := F_{n-1,1}(z) = 1/\kappa(1/z^{n-1})^{1/(n-1)},$$

its homotopy disk $\Delta(S_{F_{n-1}})$ in \mathbf{T} is of Teichmüller type. Together with estimate (4.2), this implies that for any $m > 2$ and small $r > 0$,

$$|J_n(\kappa_{m,r})| < r(n-1)^2;$$

thus

$$|J_n(\kappa_{m,\theta})| < (n-1)^2 = J_n(\kappa),$$

completing the proof of Theorem 1.2.

Remark. The above arguments work well also in the case of functionals obtained by suitable perturbation of $J_n(f)$. For example, one can take

$$J(f) = a_n^2 - a_{2n-1} + P(a_3, \dots, a_{2n-2}),$$

where P is a homogeneous polynomial of degree $2n-2$,

$$P(a_3, \dots, a_{2n-2}) = \sum_{|k|=2n-2} c_{k_3, \dots, k_n} a_3^{k_2} \dots a_n^{k_{2n-2}},$$

and $|k| := k_3 + \dots + k_{2n-2}$, $a_j = a_j(f)$, assuming that this polynomial has nonnegative coefficients and satisfies

$$\max_S |P(a_3, \dots, a_{2n-2})| < \frac{(n-1)^2}{2}.$$

For any such functional, only the Koebe function is extremal.

5. SOME NEW DISTORTION THEOREMS FOR HIGHER COEFFICIENTS

As was mentioned, Theorem 1.1 provides various new distortion estimates. For example, we obtain the following generalization of the inequality $|a_2^2 - a_3| \leq 1$ to higher coefficients.

Theorem 5.1. *For all $f \in S$ and integers $n > 3$ and $p \geq 1$,*

$$|a_n^p - a_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p.$$

This bound is sharp, and the equality only occurs for the Koebe function κ_θ .

Proof. Since $b_0 = -a_2$, the relation (1.3) yields

$$I_n(f) := a_n - a_2^{n-1} = (n-2)(-1)^{n-1}b_1b_0^{n-3} + \text{lower terms with respect to } b_0.$$

This functional satisfies the assumptions of Theorem 1.1. The same arguments as in the proof of Theorem 1.2 imply

$$|I_n(\kappa_{m,\theta})| < |I_n(\kappa_\theta)| \quad \text{for all } m \geq 2,$$

completing the proof.

In the same way, one obtains

Theorem 5.2. *For all $f \in S$ and integers $n > 2$ and $p \geq 1$,*

$$|a_{n+1}^p - a_2^p a_n^p| \leq 2^p n^p - (n+1)^p,$$

with equality only for $f = \kappa_\theta$.

REFERENCES

- [Be1] L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113-134.
- [Be2] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89-126.
- [BT] J. Brown and A. Tsao, *On the Zalcman conjecture for starlike and typically real functions*, Math. Z. **191** (1986), 467-474.
- [DB] L. de Branges *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137-152.
- [Di] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
- [EE] C.J. Earle and J.J. Eells, *On the differential geometry of Teichmüller spaces*, J. Analyse Math. **19** (1967), 35-52.
- [EKK] C.J. Earle, I. Kra and S.L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Trans. Amer. Math. Soc. **944** (1994), 927-948.
- [EM] C.J. Earle and S. Mitra, *Variation of moduli under holomorphic motions*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), Contemp. Math. **256**, Amer. Math. Soc., Providence, RI, 2000, pp. 39-67.
- [GL] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Amer. Math. Soc., 2000.
- [Go] G.M. Goluzin, *Geometric Theory of Functions of Complex Variables*, Transl. of Math. Monographs, vol. 26, Amer. Math. Soc., Providence, RI, 1969.
- [Gr] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45** (1939), 29-61.
- [Ha] W.K. Hayman, *Multivalent Functions*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 48, Cambridge University Press, 1958.
- [Je] J. A. Jenkins, *An extension of the general coefficient theorem*, Trans. Amer. Math. Soc. **95** (1960), 387-407.

- [Ko] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer, New York, 1998.
- [Kr1] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Wiley, New York, 1979.
- [Kr2] S.L. Krushkal, *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650-660.
- [Kr3] S.L. Krushkal, *Extension of conformal mappings and hyperbolic metrics*, Siberian Math. J. **30** (1989), 730-744.
- [Kr4] S.L. Krushkal, *Univalent functions and holomorphic motions*, J. Analyse Math. **66** (1995), 253-275.
- [Kr5] S.L. Krushkal, *Exact coefficient estimates for univalent functions with quasiconformal extension*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **20** (1995), 349-357.
- [Kr6] S.L. Krushkal, *Plurisubharmonic features of the Teichmüller metric*, Publications de l'Institut Mathématique-Beograd, Nouvelle série **75(89)** (2004), 119-138.
- [Kr7] S.L. Krushkal, *Quasiconformal extensions and reflections*, Ch. 11 in: Handbook of Complex Analysis: Geometric Function Theory, Vol. II (R. Kühnau, ed.), Elsevier Science, Amsterdam, 2005, pp. 507-553.
- [Kr8] S.L. Krushkal, *Proof of the Zalcman conjecture for initial coefficients*, Georgian Math. J. **17** (2010), 663-681.
- [KK1] S.L. Krushkal and R. Kühnau, *Quasikonforme Abbildungen - neue Methoden und Anwendungen*, Teubner-Texte zur Math., Bd. 54, Teubner, Leipzig, 1983.
- [KK2] S.L. Krushkal and R. Kühnau, *Quasiconformal reflection coefficient of level lines*, Complex Analysis and Dynamical Systems IV, Contemporary Mathematics (2011), to appear.
- [Ku1] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77-105.
- [Ku2] R. Kühnau, *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q -quasikonforme Fortsetzbarkeit?*, Comment. Math. Helv. **61** (1986), 290-307.
- [Ma] W. Ma, *The Zalcman conjecture for close-to-convex functions*, Proc. Amer. Math. Soc. **104** (1988), 741-744.
- [Po] Chr. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
- [Ro] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces (Ann. of Math. Stud. vol. 66), Princeton Univ. Press, Princeton, 1971, pp. 369-383.
- [Sh] Y.L. Shen, *Pull-back operators by quasisymmetric functions and invariant metrics on Teichmüller spaces*, Complex Variables **42** (2000), 289-307.
- [St] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Analyse Math. **30** (1976), 464-480.
- [Ve] E. Vesentini, *Complex geodesics and holomorphic mappings*, Sympos. Math. **26** (1982), 211-230.
- [Zh] I.V. Zhuravlev, *Univalent functions and Teichmüller spaces*, Inst. of Mathematics, Novosibirsk, preprint, 1979, 1-23 (Russian).

Department of Mathematics, Bar-Ilan University
 52900 Ramat-Gan, Israel
 and Department of Mathematics, University of Virginia,
 Charlottesville, VA 22904-4137, USA